Quantum analysis of the z-scan technique

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A Gaussian-wave theory is developed for the classical and quantum analysis of the z-scan method that is often used to measure third-order nonlinearities. The theory allows us to compute the transmittance in the z scan and the associated regimes of amplitude squeezing. The classical limits of our theory are in perfect agreement with the previous theoretical results. We show that amplitude squeezing of 1.2 dB can be obtained using the z scan with a careful selection of the signal power and the aperture size.

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1. INTRODUCTION

Generation of amplitude-squeezed light may be achieved by filtering light after its propagation through a medium with a cubic nonlinearity.[1–5] The light may be filtered either spectrally[1–4] or spatially.[5] Due to the nonlinearity of the medium, a portion of the light gets squeezed whereas the remaining portion is antisqueezed. The overall effect of the nonlinearity on the input light does not change the mean number of photons or the variance, however, one can obtain amplitude-squeezed light by carefully selecting the squeezed portion.

A similar technique, which is called the z scan,[6,7] has been efficiently used for the measurement of cubic nonlinearities. In the method of the z scan, the transmittance of a nonlinear medium through a finite aperture in the far field is measured as a function of the sample position. The transmittance function not only gives immediate information about the sign of the nonlinearity, but it also allows one to easily estimate the magnitude of the nonlinearity.

In this paper, we introduce a new model for the classical and quantum analysis of the z-scan technique. We assume a lossless medium with a cubic nonlinearity, and a Gaussian beam is used as a signal. Our theory is valid for weak focusing, where the interaction length is much smaller than the confocal parameter of the signal beam and allows us to compute not only the transmittance function but also the amplitude squeezing obtainable after the aperture in the far field. We show that an amplitude squeezing of 1.2 dB can be obtained using the z-scan technique.

2. CLASSICAL ANALYSIS OF THE z SCAN

In this section, we briefly describe the technique of the z scan and formulate a method for its classical analysis. This method allows us to compute the transmittance function in a different way than the previous methods.[6,7] Furthermore, in Section 3, we show that this method, unlike previous methods, allows us to compute the amount of amplitude squeezing at the output of the z-scan setup.

Figure 1 shows a typical z-scan setup for measuring the third-order nonlinearity of a medium. The input beam is focused onto the sample with a lens, and the resulting output of the sample is passed through an aperture in the far field. After passing through the aperture, the power of the beam is measured with a detector (D2). Because the sample has a third-order nonlinearity, it will have positive (negative) lensing effect for positive (negative) cubic nonlinearities, and as a result, the measured power will be dependent on the position of the sample. A small portion of the input beam is split using a beam splitter and measured by a detector (Dl) to be used as a reference beam. The transmittance function, which is the ratio of D2 to Dl, is recorded as the sample is moved along the z axis. Using the transmittance function, one can determine the sign and the amplitude of the third-order nonlinearity of the sample.

The method we formulate in this section assumes a Gaussian beam at the input and a lossless medium. Our analysis is valid for weak focusing, i.e., the confocal parameter of the Gaussian beam is much greater than the length of the sample.

Because of the cylindrical symmetry of the z-scan setup, we express the signal field as a $+z$ propagating wave, whose transverse amplitude depends only on the radial distance ρ. Therefore, the signal field is described by

$$E_s(\mathbf{r},t) = \frac{1}{2}A_s(\rho,z)\exp[i(k_s z - \omega t)] + \text{c.c.}, \tag{1}$$

where ω is the angular frequency and c.c. denotes the complex conjugate of the first term. Under the slowly varying envelope approximation, evolution of the signal field in a medium with cubic nonlinearity is governed by[8]

$$\frac{\partial A_s(\rho,z)}{\partial z} + \frac{1}{2ik_s}\nabla_\perp^2 A_s(\rho,z) = i\frac{n_2\omega}{c}|A_s(\rho,z)|^2 A_s(\rho,z), \tag{2}$$

where k_s is the wave-vector magnitude, A_s is the complex-field amplitude of the signal field, and n_2 is the nonlinear index of the medium.

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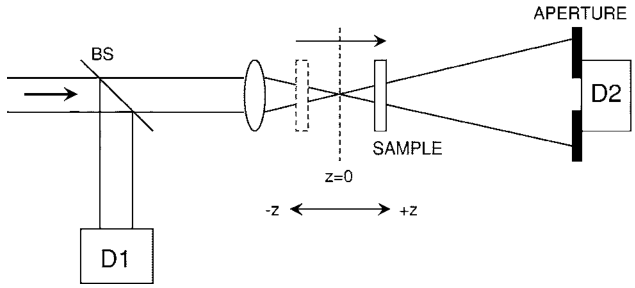

Fig. 1. Typical z-scan setup for measuring the third-order nonlinearity.

When there is no nonlinear interaction, Eq. (2) with $n_2=0$ gives the following solution for the evolution of a Gaussian signal beam:

$$A_s(\rho,z)=\frac{A_{s0}}{1+i2z/z_0}\exp\left(\frac{-\rho^2/w_0^2}{1+i2z/z_0}\right), \tag{3}$$

where A_{s0} is the on-axis peak amplitude of the signal field at $z=0$, w_0 is the beam waist, and $z_0=k_s w_0^2$ is the confocal parameter (twice the Rayleigh range). Using the normalizations $\rho=rw_0/\sqrt{2}$ and $z=\xi z_0$, Eqs. (2) and (3) become

$$\frac{\partial A_s(r,\xi)}{\partial \xi}-i\nabla_\perp^2 A_s(r,\xi)=i\frac{n_2\omega z_0}{c}|A_s(r,\xi)|^2A_s(r,\xi), \tag{4}$$

$$A_s(r,\xi)=\frac{A_{s0}}{1+i2\xi}\exp\left(\frac{-r^2/2}{1+i2\xi}\right). \tag{5}$$

As shown in Fig. 1, in the z-scan technique, the transmittance of the signal field through a finite aperture placed in the far field is measured as the sample position z is varied. For a complete classical analysis of the z-scan technique, Eq. (4) should be solved for different positions of the nonlinear medium. If the position of the nonlinear sample of length l is z_c with respect to the beam waist at $z=0$, then Eq. (4) should be solved for $\xi_{\text{in}}<\xi<\xi_{\text{out}}$, where

$$\xi_{\text{in}}=(z_c-l/2)/z_0, \tag{6}$$

$$\xi_{\text{out}}=(z_c+l/2)/z_0. \tag{7}$$

The $|A_s(r,\xi)|^2$ term on the right-hand side of Eq. (4) accounts for the self-phase modulation of the Gaussian beam inside the nonlinear medium. Due to the (r,ξ) dependence, each part of the signal beam experiences a different phase shift. However, in the weak focusing regime, $l/z_0 \ll 1$, where the effect of diffraction over the length of the medium is negligible, this term can be replaced by

$$|A_s(r,\xi)|^2=\frac{|A_{s0}|^2}{1+4\xi_c^2}\exp\left(\frac{-r^2}{1+4\xi_c^2}\right), \tag{8}$$

where $\xi_c=z_c/z_0$. In the case of tight focusing, where the length of the nonlinear medium becomes comparable to the confocal parameter ($l\sim z_0$), the signal beam propagates differently, and Eq. (8) will no longer be valid. Inserting Eq. (8) into Eq. (4), we get

$$\frac{\partial A_s(r,\xi)}{\partial \xi}-i\nabla_\perp^2 A_s(r,\xi)=\frac{i\gamma}{1+4\xi_c^2}\exp\left(\frac{-r^2}{1+4\xi_c^2}\right)A_s(r,\xi), \tag{9}$$

where

$$\gamma=\frac{n_2\omega z_0}{c}|A_{s0}|^2. \tag{10}$$

Using the intensity-dependent refractive index n_2^I instead of n_2 and writing $|A_{s0}|^2$ in terms of the signal power P, γ can be rewritten as

$$\gamma=n_2^I\frac{P}{4\pi\lambda^2}. \tag{11}$$

The way we solve Eq. (4) is similar to the method that is used in Refs. 9 and 10. If the right-hand side of Eq. (4) was zero, one obtains the paraxial Helmholtz equation. In the cylindrically symmetric geometry, we are dealing with, the general solution of the paraxial Helmholtz equation can be expressed as a sum of the Laguerre–Gaussian modes:

$$A_s(r,\xi)=\sum_{n=0}^{\infty}A_n(\xi)G_n(r,\xi), \tag{12}$$

where

$$G_n(r,\xi)=L_n\left(\frac{r^2}{1+4\xi^2}\right)\frac{1}{1+i2\xi}\exp\left(\frac{-r^2/2}{1+i2\xi}\right)\times\exp(-i2n\tan^{-1}2\xi) \tag{13}$$

is the nth cylindrically symmetric Laguerre–Gaussian mode with L_n being the nth Laguerre polynomial, and A_n is the complex amplitude of the nth mode. The modes defined in Eq. (12) form an orthonormal set, where

$$2\int_0^\infty G_n(r,\xi)G_m^*(r,\xi)r\mathrm{d}r=\delta_{mn}. \tag{14}$$

The orthonormality property allows us to compute the projection of the output signal field on each of the modes. By doing so, we obtain an equation for the evolution of the mode amplitude $A_n(\xi)$, thus converting Eq. (4) into a set of ordinary differential equations in matrix form.[9] If we insert Eq. (12) into Eq. (4) and use the orthonormality property of the modes, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}A=i\frac{\gamma}{1+4\xi_c^2}TA, \tag{15}$$

where A is a signal column vector whose transpose A^T has elements,

$$A^T(\xi)=[A_0(\xi)\quad A_1(\xi)\quad A_2(\xi)\quad \cdots], \tag{16}$$

and T is a square matrix with elements,

$$T_{mn} = \left[\frac{2^{-m-n-1}(m+n)!}{m\,!\,n!} \right] \exp(-i2(n-m)\tan^{-1} 2\xi_c). \tag{17}$$

Equation (15) is an ordinary differential equation for the column vector A, whose solution is

$$A(\xi_{\text{out}}) = \exp\left(i \frac{\Phi_{\text{nl}}}{1+4\xi_c^2} T \right) A(\xi_{\text{in}}), \tag{18}$$

where

$$\Phi_{\text{nl}} = \gamma \frac{l}{z_0} \tag{19}$$

is the nonlinear phase shift at the center of the Gaussian beam when $\xi_c=0$. Since only the fundamental Gaussian beam is present at the input, the initial condition is

$$A(\xi_{\text{in}}) = [1 \quad 0 \quad 0 \quad \cdots]^T. \tag{20}$$

The aim of our analysis is to compute the transmittance of the aperture placed in the far field (see Fig. 1). The transmittance function is defined as the power transmitted through the aperture normalized with respect to the power of a reference beam. The reference beam is usually chosen as the beam that could pass through the aperture when the nonlinear medium is removed. With this description, the transmittance function can be written as

$$t = \frac{\int_0^{r_a} |A_s(r,\xi_d)|^2 r \mathrm{d}r}{\int_0^{r_a} |G_0(r,\xi_d)|^2 r \mathrm{d}r}, \tag{21}$$

where r_a is the radius of the aperture centered on the Gaussian beam and ξ_d is the normalized axial coordinate of the aperture. The beam radius of the Gaussian beam increases as we go further away from the sample. Therefore, from an experimental point of view, it is convenient to define a new unitless quantity for the aperture size,

$$S = \frac{r_a}{\sqrt{2(1+4\xi_d^2)}}, \tag{22}$$

which is equal to the ratio of the aperture radius to the beam radius of the Gaussian beam at the plane of detection. Since the aperture is placed far from the beam waist of the Gaussian beam, we can assume that the Gouy phase of the fundamental beam at the aperture is

$$\tan^{-1}(2\xi_d) = \pi/2. \tag{23}$$

Using Eqs. (12), (21), and (23), one can prove that for the limiting case where the aperture radius goes to zero ($S \to 0$), the transmittance function is

$$t = \left| \sum_{n=0}^{\infty} (-1)^n A_n(\xi_{\text{out}}) \right|^2. \tag{24}$$

In Fig. 2, we demonstrate the results of our calculations for different values of the nonlinear phase shifts wherein we have assumed a positive nonlinear refractive index ($n_2>0$). In such a case, the nonlinear medium gives a positive lensing effect. For $z<0$, the positive lensing effect brings the focus closer to the sample, i.e., further away from the aperture and, therefore, a decrease in the transmittance is observed. For $z>0$, however, the nonlinear medium tends to collimate the beam, leading to an increased transmittance. Experimentally, the difference between the maximum and the minimum of the transmittance function gives information about the value of γ. By knowing the power of the beam, we can then calculate the nonlinear refractive index using Eq. (10).

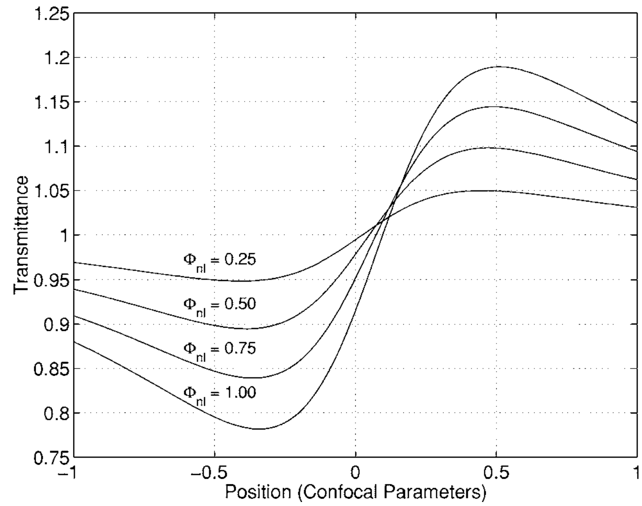

Fig. 2. Transmittance as a function of the position of the nonlinear medium for different values of the nonlinear phase shift.

A classical analysis of the z-scan method was first carried out by Sheik-Bahae *et al.*[6,7] In that analysis, the authors used the method of "Gaussian decomposition" given by Weaire *et al.*[11] The modes in this analysis are not orthonormal, unlike the Laguerre–Gaussian modes used in our theoretical model. The orthonormality of the modes makes it easier to compute the quantum effects, such as amplitude squeezing as we will demonstrate in the following section. We checked the accuracy of our classical results above by comparing them with those in Ref. 7 and found that they were in perfect agreement.

Equation (24) gives the transmittance function for an aperture with an infinitesimally small radius. To calculate the transmittance for a finite radius, one has to compute the field $A_s(r,\xi_d)$ using the mode amplitudes and then calculate the integral in Eq. (21) numerically. Using the formulation we have developed above, we analyze the response of the z-scan setup for different values of S. Figure 3 shows the results of such an analysis, where the output power is plotted as a function of the input power for different values of S, while ξ_c is held fixed at −0.5. Both axes are normalized with respect to P_0, which is the amount of power that gives a nonlinear phase shift of 1 rad ($\Phi_{\text{nl}}=1$). The relation between the output and input powers is

$$P_{\text{out}} = [1-\exp(-2S^2)]t(\Phi_{\text{nl}},S)P_{\text{in}}, \tag{25}$$

where the first term accounts for the loss of power due to the aperture in the absence of the nonlinearity. For no aperture ($S=\infty$), the input and output powers are equal. For an aperture of finite radius, the response is nonlinear, an

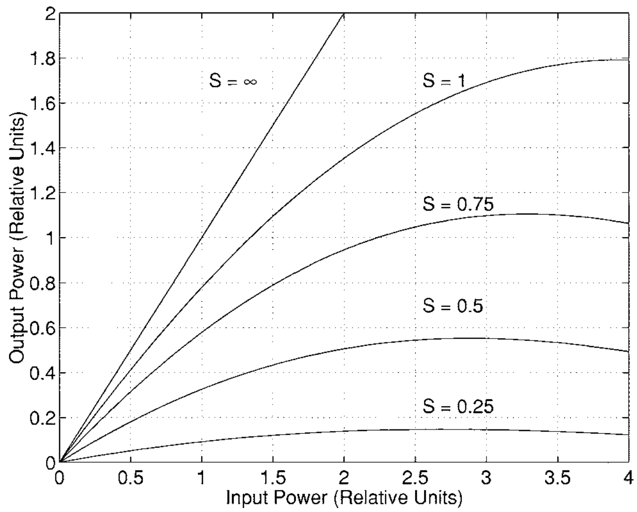

Fig. 3. Output power versus input power for different values of the normalized aperture radius ($\xi_c=-0.5$). Both axes are normalized with respect to P_0, which is the amount of power that gives a nonlinear phase shift of 1 rad ($\Phi_{\mathrm{nl}}=1$).

effect introduced by the cubic nonlinearity of the medium.

Now we consider the effect of the nonlinear transmittance of the aperture on fluctuations in the incident field. Classically, the fluctuations in the output of the z-scan setup are related to the fluctuations in the input through the derivative $\mathrm{d}P_{\mathrm{out}}/\mathrm{d}P_{\mathrm{in}}$. In case of a linear relationship between P_{out} and P_{in}, the noise characteristics are unchanged since

$$\frac{\mathrm{d}P_{\mathrm{out}}}{\mathrm{d}P_{\mathrm{in}}}=\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}. \tag{26}$$

However, one should expect reduced noise at the output when

$$\left|\frac{\mathrm{d}P_{\mathrm{out}}}{\mathrm{d}P_{\mathrm{in}}}\right|<\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}, \tag{27}$$

i.e., the noise transfer to the output is less than the power transfer. Obviously, the point with the zero derivative becomes most important, because it would give the minimum output noise. However, one must note that at this point, the contribution from higher-order derivatives will be effective; therefore, the output noise will not be zero.

Although the above is a classical approach, it outlines the main reason why one may expect squeezing in a z-scan setup, in particular, when working with a transparent third-order (cubic) nonlinear medium, i.e., a medium exhibiting negligible linear and nonlinear absorption and possessing only a dispersive nonlinearity. In Section 3, below, we present a quantum analysis of the z-scan setup for a medium with such dispersive nonlinearity,[12] which predicts that amplitude-squeezed light can be generated if the position of the medium and the size of the aperture are carefully selected.

3. QUANTUM ANALYSIS OF THE z SCAN

The z-scan technique has some important similarities with previously proposed setups for the generation of squeezing in media with cubic nonlinearities. One of these techniques is spectral filtering of solitons.[1] In this technique, after their propagation in a fiber, the solitons are spectrally filtered, and the noise properties of the resulting light shows regimes of amplitude squeezing. In the z-scan technique, propagation of the beam after passing through the nonlinear medium similarly gives rise to the Fourier transform of the input beam ($z=0$) in the far field. The aperture then acts as a filter that selects the spatial frequencies in a certain band. Therefore, one can expect to observe amplitude squeezing in a z-scan setup.

For a complete quantum analysis of the z-scan technique, one has to solve the quantum version of Eq. (4), which is[13–16]

$$\frac{\partial \hat{A}_s(r,\xi)}{\partial \xi}-i\nabla_\perp^2\hat{A}_s(r,\xi)=i\frac{n_2\omega z_0}{c}\hat{A}_s^\dagger(r,\xi)\hat{A}_s(r,\xi)\hat{A}_s(r,\xi), \tag{28}$$

where $\hat{A}_s$ is the quantized field amplitude of the signal. We apply the linearization approximation and expand the quantized field amplitude as a sum of its mean field and the associated fluctuation operator as

$$\hat{A}_s=A_s+\Delta\hat{A}_s, \tag{29}$$

where $A_s=\langle\hat{A}_s\rangle$ and the mean of the fluctuation operator is zero, i.e., $\langle\Delta\hat{A}_s\rangle=0$. By inserting Eq. (29) into Eq. (28), one finds that the evolution of the mean field A_s is governed by Eq. (4), which we have solved in Section 2, and the fluctuation operator $\Delta\hat{A}_s$ obeys a linear equation[5]

$$\frac{\partial \Delta\hat{A}_s(r,\xi)}{\partial \xi}-i\nabla_\perp^2\Delta\hat{A}_s(r,\xi)=i2\frac{n_2\omega z_0}{c}|A_s|^2\Delta\hat{A}_s(r,\xi)+i\frac{n_2\omega z_0}{c}(A_s)^2\Delta\hat{A}_s^\dagger(r,\xi). \tag{30}$$

On the right-hand side of Eq. (30), we only retain the first-order fluctuation terms and neglect the higher-order terms. This approximation requires that the mean field is much greater than the fluctuations. Furthermore, the linearization approximation can break down for large interaction distances if there is an exponential growth in the fluctuations.

The mean-field quantities in the above equation can be obtained from the solution in Section 2. The term $|A_s|^2$ is already given in Eq. (8). Furthermore, since we are trying to solve the z-scan problem for a thin medium ($l/z_0\ll 1$), the mean field at the output consists of the mean field at the input multiplied by a nonlinear phase term that is dependent on the intensity. Therefore, the term $(A_s)^2$ in Eq. (30) is

$$(A_s(r,\xi))^2=|A_s(r,\xi)|^2\exp\left(\frac{i2n_2\omega z_0(\xi-\xi_{\mathrm{in}})}{c}|A_s(r,\xi)|^2\right). \tag{31}$$

To solve Eq. (30), once again we apply the modal expansion to the fluctuation operator. We can express $\Delta\hat{A}_s$ as

$$\Delta\hat{A}_s(r,\xi) = \sum_{n=0}^{\infty} \Delta\hat{A}_n(\xi) G_n(r,\xi), \tag{32}$$

where $\Delta\hat{A}_n$ is the fluctuation operator corresponding to the nth Laguerre–Gaussian mode. Inserting Eq. (32) into Eq. (30) and using Eqs. (8) and (31), we can write the evolution of the fluctuation operators in a matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Delta\hat{A} = i2\frac{\gamma}{1+4\xi_c^2}T\Delta\hat{A} + i\frac{\gamma}{1+4\xi_c^2}U\exp\left(i2\frac{\gamma(\xi-\xi_{\mathrm{in}})}{1+4\xi_c^2}T\right)\Delta\hat{A}^{\dagger}, \tag{33}$$

wherein $\Delta\hat{A}$ is a vector whose components are the fluctuation operators of the modes and U is a matrix, the components of which are

$$U_{mn} = \left[\frac{2^{-m-n-1}(m+n)!}{m!\,n!}\right]\exp(i2(m+n)\tan^{-1}2\xi_c). \tag{34}$$

Equation (33) can be solved more easily if we define a new vector $\Delta\hat{A}'$ as

$$\Delta\hat{A}' = \exp\left(-i\frac{\gamma(\xi-\xi_{\mathrm{in}})}{1+4\xi_c^2}T\right)\Delta\hat{A}. \tag{35}$$

Inserting Eq. (35) into Eq. (33) we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Delta\hat{A}' = i\frac{\gamma}{1+4\xi_c^2}T\Delta\hat{A}' + i\frac{\gamma}{1+4\xi_c^2}U\Delta\hat{A}'^{\dagger}. \tag{36}$$

The solution of Eq. (36) may be written in the form

$$\Delta\hat{A}'(\xi) = M'(\xi)\Delta\hat{A}'(\xi_{\mathrm{in}}) + N'(\xi)\Delta\hat{A}'^{\dagger}(\xi_{\mathrm{in}}), \tag{37}$$

where M' and N' are state transition matrices. By inserting Eq. (37) into Eq. (36) and taking the second derivatives of the state transition matrices, we end up with

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}M' = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}N' = 0. \tag{38}$$

Using the initial conditions,

$$M'(\xi_{\mathrm{in}}) = I, \tag{39}$$

$$N'(\xi_{\mathrm{in}}) = 0, \tag{40}$$

where I is the identity matrix, the state transition matrices at the output are found to be

$$M'(\xi_{\mathrm{out}}) = I + i\frac{\Phi_{nl}}{1+4\xi_c^2}T, \tag{41}$$

$$N'(\xi_{\mathrm{out}}) = i\frac{\Phi_{nl}}{1+4\xi_c^2}U. \tag{42}$$

Using Eqs. (35), (37), (41), and (42) we get

$$\Delta\hat{A}(\xi_{\mathrm{out}}) = M\Delta\hat{A}(\xi_{\mathrm{in}}) + N\Delta\hat{A}^{\dagger}(\xi_{\mathrm{in}}), \tag{43}$$

where

$$M = \exp\left(i\frac{\Phi_{nl}}{1+4\xi_c^2}T\right)\left(I + i\frac{\Phi_{nl}}{1+4\xi_c^2}T\right), \tag{44}$$

$$N = \exp\left(i\frac{\Phi_{nl}}{1+4\xi_c^2}T\right)\left(i\frac{\Phi_{nl}}{1+4\xi_c^2}U\right). \tag{45}$$

Although the only field component with nonzero mean at the input of the z-scan setup is the fundamental mode, the noise contribution comes from all modes at the input. Even when there is no mean field in a particular mode, there are vacuum fluctuations. Since photons are bosons, the modal fluctuation operators at the input face of the nonlinear medium are

$$\Delta\hat{A}_n(\xi_{\mathrm{in}}) = \hat{c}_n, \tag{46}$$

where $\{\hat{c}_n\}$ are the vacuum-state operators with zero mean and obeying the bosonic commutation relation $[\hat{c}_m, \hat{c}_n^{\dagger}] = \delta_{mn}$.

In the experimental configuration shown in Fig. 1, the average photocurrent and the photocurrent fluctuations are measured using direct detection. In such detection, the two quantities are proportional to the mean and the variance of the photon number arriving in some detection time. The photon number operator that is measured by such a detection scheme is

$$\hat{n} = \int_0^{\infty} \hat{A}_s^{\dagger}(r,\xi)\hat{A}_s(r,\xi) r\mathrm{d}r. \tag{47}$$

Using Eqs. (12), (29), and (32), and the orthonormality property of the Laguerre–Gaussian beams, one can show that

$$\langle\hat{n}\rangle \simeq \sum_{n=0}^{\infty} |A_n|^2, \tag{48}$$

$$\Delta\hat{n} \simeq \sum_{n=0}^{\infty} A_n\Delta\hat{A}_n^{\dagger} + A_n^{*}\Delta\hat{A}_n. \tag{49}$$

Inserting Eq. (43) into Eq. (49), we get

$$\Delta\hat{n} = \sum_{n=0}^{\infty}\left[\left(\sum_{m=0}^{\infty} q_m^{*}M_{mn} + q_m N_{mn}^{*}\right)\hat{c}_n + \left(\sum_{m=0}^{\infty} q_m^{*}N_{mn} + q_m M_{mn}^{*}\right)\hat{c}_n^{\dagger}\right], \tag{50}$$

where $\{q_m\}$ are the complex field amplitudes at the output. The Fano factor of the output beam is then found to be

$$F = \frac{\langle(\Delta\hat{n})^2\rangle}{\langle\hat{n}\rangle} = \frac{\sum_{n=0}^{\infty}|\sum_{m=0}^{\infty} q_m^{*}M_{mn} + q_m N_{mn}^{*}|^2}{\sum_{n=0}^{\infty}|q_n|^2}. \tag{51}$$

If no aperture is used in the detection plane, the weight coefficients will be equal to the field amplitudes [$q_n = A_n(\xi_{\mathrm{out}})$]. In such a case, the Fano factor at the output can be calculated using Eqs. (18), (41), and (42), and it is found to be unity. However, when an aperture is used, the weight coefficients should be calculated by expanding the field after the aperture into the Laguerre–Gaussian beams. If a circular aperture is used, where the center of the aperture is the same as the center of the Gaussian beam, the new weight coefficients can be calculated as

$$q_n = \int_0^{r_a} A_s(r,\xi_d) G_n^*(r,\xi_d) r \mathrm{d}r. \tag{52}$$

The weight coefficients for different values of S can be calculated numerically using Eqs. (13) and (52). In our analysis, we compute the weight coefficients for different aperture sizes using finite difference techniques. We calculate the integral in Eq. (52) for the first 20 Laguerre–Gaussian modes.

Figure 4 shows the results of our simulations for the calculation of the Fano factor at the output where $\Phi_{\mathrm{nl}}=1$ and $\xi_c=0.25$. For $S=0$, the Fano factor is unity. As we increase the aperture size, the Fano factor starts to increase and reaches a maximum of 0.15 dB at $S=0.62$. When the aperture size is increased further, the Fano factor decreases and reaches a minimum of -0.175 dB at $S=1.5$. For larger values of S, the Fano factor asymptotically reaches 0 dB, which is the expected Fano factor for the case of no aperture.

Figure 5 shows the results of our simulations for the calculation of minimum obtainable Fano factor (maximum amplitude squeezing) for $\Phi_{\mathrm{nl}}=4$. The figure also includes the normalized aperture size needed for obtaining the maximum amplitude squeezing. The minimum Fano factor for this nonlinear phase shift is -1.2 dB, which occurs at $z/z_0=-0.95$, and the required normalized aperture radius is 1.05. We also ran the same simulation for different values of the nonlinear phase shifts. Figure 6 shows the results for such a simulation, where we plot the minimum Fano factor as a function of the nonlinear phase shift. The position of the nonlinear medium and the aperture radius for achieving this Fano factor are also included in the figure. The position of the sample for achieving maximum amplitude squeezing is always negative, i.e., the nonlinearity should be placed before the focus of the Gaussian beam. The normalized aperture radius stays almost constant at $S=1.05$. That is, the radius of the aperture should be slightly larger than the beam radius of the signal for maximum squeezing. For $\Phi_{\mathrm{nl}}=4$, 1.2 dB of amplitude squeezing can be generated with a careful selection of the position and radius of the aperture.

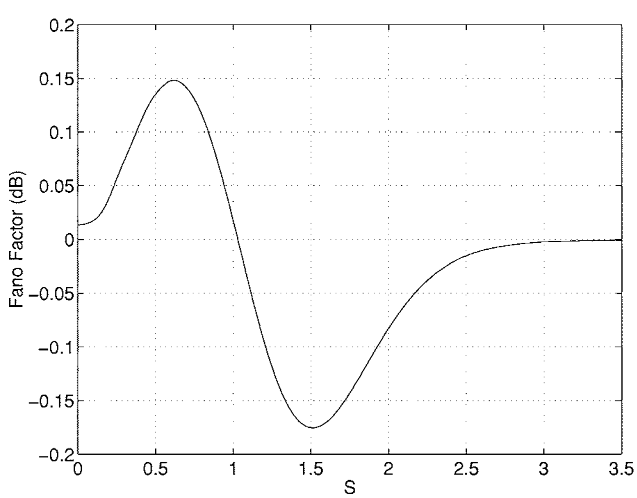

Fig. 4. Fano factor in decibels as a function of the aperture radius for $\Phi_{\mathrm{nl}}=1$ and $\xi_c=0.25$.

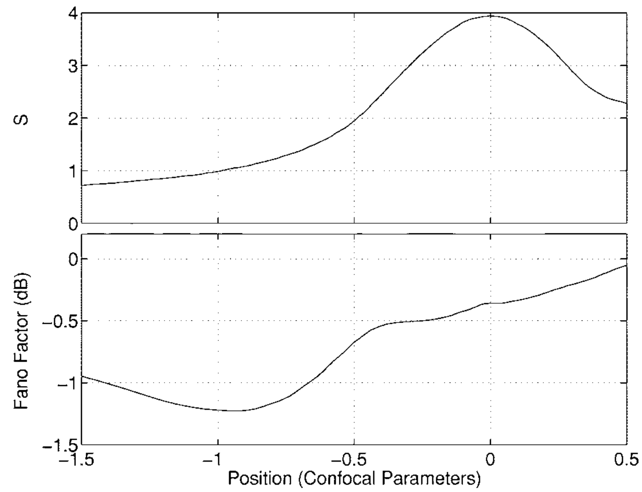

Fig. 5. Minimum Fano factor and the required aperture radius as a function of the position of the nonlinear medium ($\Phi_{\mathrm{nl}}=4$).

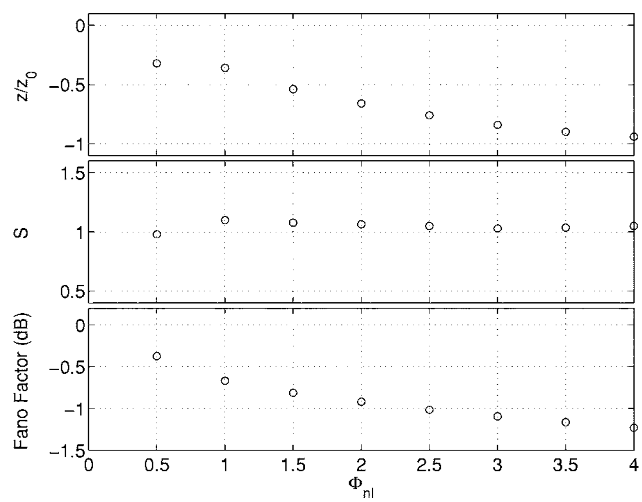

Fig. 6. Minimum Fano factor, required aperture radius, and required position of the nonlinear sample as a function of the nonlinear phase shift Φ_{nl}.

The results for negative nonlinearities are exactly the same except that they are all inverted around $\xi_c=0$. The general structure of both the classical and quantum analyses remain the same, and only the sign of the position of the medium changes. As an example, for $\Phi_{\mathrm{nl}}=-4$, the maximum squeezing will be 1.2 dB, which occurs at $z/z_0=0.95$ and the required normalized aperture radius is 1.05.

The main source of error in our calculations is caused by the numerical calculation of the weight coefficients. Although the state transition matrices M and N can be calculated analytically, the weight coefficients at the output should be calculated using numerical integration methods. Furthermore, only a finite number of modes can be calculated with this method, and higher-order modes are neglected. The Laguerre modes for $n>20$ are difficult to compute accurately because of the limitations caused by numerical precision. This causes problems especially for larger aperture sizes. Within the boundaries of parameters that are investigated in this paper, the total numerical error is less than 1%.

4. CONCLUSION

In this paper, we have shown theoretically that amplitude squeezing is obtainable with a typical z-scan setup, which entails propagation through cubic dispersive nonlinearity followed by passage through an aperture, i.e., spatial filtering. Previously, it has been demonstrated that amplitude squeezing can be achieved by propagating through an optical fiber, which possesses such a nonlinearity, followed by spectral filtering. The two cases are, in fact, analogous since the equations governing diffraction and propagation through a fiber are similar except for the difference of dimensionality in the transverse terms. Our theoretical model assumes that the input beam is Gaussian, and the length of the nonlinear medium is much smaller than the confocal length of the beam. For such a case, we have analytically solved the problem of propagation through the cubic nonlinearity; however, numerical methods have been used to calculate the mode coefficients after the filtering. In our analysis, the errors introduced by the numerical methods and the finite number of modes used are less than 1%. We have shown that for a nonlinear phase shift of $\Phi_{nl}=4$, 1.2 dB amplitude squeezing is obtainable.

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